Eur. Phys. J. C (2008) 56: 483–492
DOI 10.1140/epjc/s10052-008-0673-8

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Chiral susceptibility and chiral phase transition in Nambu–Jona-Lasinio model

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Received: 5 March 2008 / Revised: 29 May 2008 / Published online: 9 August 2008

Abstract We give a general relation between the chiral susceptibility and the thermodynamical potential and a relation between the chiral susceptibility and the condition for furcations to appear in the Wigner solution(s) in the Nambu–Jona-Lasinio (NJL) model. We find that the chiral susceptibility is a quantity able to represent the appearance of furcation in the solution(s) of the gap equation and the concavo–convexity of the thermodynamical potential in the NJL model. It indicates that the chiral susceptibility can identify the stability of the states and the chiral phase transition in NJL model. We propose that analyzing the chiral susceptibility may play an important role in studying the chiral phase transition in approaches superior to the NJL model.

PACS 12.38.Aw · 05.70.Fh · 11.30.Rd · 12.40.-y

1 Introduction

Quantum chromodynamics (QCD) is a non-Abelian gauge theory. The proof of its renormalizability [1] and the discovery of ultraviolet asymptotic freedom [2, 3] have been milestones in its acceptance as the theory of the strong interaction. For large momentum, the coupling becomes very weak; then perturbation theory is appropriate to carry out the calculations. However, for small momentum, the coupling grows quite strong and adequate methods have to be implemented to study non-perturbative phenomena, such as confinement, dynamical chiral symmetry breaking (DCSB) and bound state formation. Among these characteristics, DCSB is fundamentally important. For example, it is responsible for the generation of large constituent-like masses for dressed quarks in QCD, and it also is the keystone in the realization of Goldstone's theorem through pseudoscalar mesons in QCD. It is generally believed that, at sufficiently high temperature and/or density, the QCD vacuum undergoes a phase transition into a chiral symmetric phase. This chiral phase transition plays an essential role in studying the structure of some astro-objects and the evolution of the early universe, which may be experimentally realized in ultra high energy heavy-ion collisions. At finite temperature, the lattice simulation is a powerful tool for studying the chiral phase transition. It is now being developed also for a finite chemical potential. However, effective theories of QCD are still necessary, and they even are powerful for our understanding of various non-perturbative phenomena, including the phase transition. There have been many approaches and models exhibiting such a character, such as the Dyson–Schwinger equation (DSE) approach [4–25], the Nambu–Jona-Lasinio (NJL) model [26–33] and the Polyakov-loop improved NJL model [34–41], the chiral quark model [42–54], the global color symmetry model (GCM) [55–66], the quark–meson coupling (QMC) model [67–70] and the Polyakov-loop extended QMC model [71], the quark mean field model [72], and other effective field theory models (see for example Ref. [73]).

In these QCD-like theories, one usually first tries to find the solutions of the equations satisfied by the order parameters of the phase transition and then to determine which solution is stable under certain conditions by analyzing the thermodynamical potential. In Ref. [23], with the DSE approach, some of us have shown that the chiral susceptibility is a quantity which could describe the dependence of the chiral condensate or dressed quark mass on the chemical potential in the first order approximation. The chiral susceptibility χ is usually defined as the first order response of the chiral condensate or of the dressed quark mass with respect to the current quark mass. Furthermore, Ref. [23] suggests that the chiral susceptibility could be used to represent the possibility of the chiral and other phase transitions. How-

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ever, the explicit relation between the chiral susceptibility and the thermodynamical potential has not yet been given.

We study the properties of χ in this paper and discuss that it is just the right quantity to use on identifying the possibility of phase transitions.

The DSE approach and the NJL model both have a quark (gap) equation, and the NJL model can be regarded as an approximation of the DSE, since the NJL model takes the point-like interaction approximation for the gluon-mediated interaction among quarks as a starting point. Even though the DSE is an approach superior to the NJL model, finding the solutions of DSEs and determining which solution is the physical solution remain difficult problems in general, because as the higher order loops are taken into account, more equations need to be solved, and it becomes more complicated or even impossible to give the explicit expression of the thermodynamical potential. However, there are not so many difficulties in the NJL model. Thus, if we could find some quantities which may characterize the properties of the solutions in a simple model, for example the NJL model, and generalize it to more advanced models such as DSE, it will be very helpful in studying the properties of DCSB. We thus study the solutions of the gap equation and the relation between the chiral susceptibility and the thermodynamical potential in the NJL model in this paper.

The paper is organized as follows. In Sect. 2, we describe briefly the main points of the NJL model and the corresponding gap equation for the quark. In Sect. 3, we discuss the properties of the chiral susceptibility and give the relation between the chiral susceptibility and the thermodynamical potential and that between the chiral susceptibility and the condition for furcation(s) to emerge in the solution(s) of the gap equation in NJL model. In Sect. 4, we show the validity of the chiral susceptibility in identifying the chiral phase transition by numerically analyzing the characteristics of the solutions of the gap equation, the thermodynamical potential and the chiral susceptibility with respect to the chemical potential in the NJL model. Finally, in Sect. 5, we give a summary and present some remarks.

2 Brief description of the NJL model

The Nambu–Jona-Lasinio (NJL) model was originally developed as a theory to study the interaction of nucleons through an effective two-body interaction [26]. Subsequently, it was extended to a description of the interaction in the quark degrees of freedom [28, 29, 31–33]. Because of its simplicity, the NJL model is useful for us to gain an understanding of the process of the DCSB. Also it could explicitly show us the Goldstone modes.

As an effective approach, the NJL model has been used to study the chiral phase transition in matter at finite temperature and baryon chemical potential, color superconductivity at moderate baryon density, the properties of some asymmetry matter, and the structure of some astro-objects (see for example Refs. [33, 34, 36, 38, 39, 41, 74–104]). In this paper we discuss only the chiral dynamical behavior and start with the simple Lagrangian of the NJL model,

$$\mathcal{L}_{\mathrm{NJL}} = \bar{q}\left(i\gamma^{\mu}\partial_{\mu} - m\right)q + G\left[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma_5\vec{\tau}q)^2\right], \tag{1}$$

where m is the current quark mass, the $\vec{\tau}$ are Pauli matrices, and G is the coupling constant.

In the Hartree approximation, the interaction terms could be substituted as follows:

$$(\bar{q}\hat{O}q)^2 \longrightarrow 2\langle\bar{q}\hat{O}q\rangle\bar{q}\hat{O}q - \langle\bar{q}\hat{O}q\rangle^2,$$

in which $\hat{O}$ could be the interaction matrices 1, γ_5, γ_μ and $\gamma_5\gamma_\mu$. If we only take the scalar condensation into account, the Lagrangian might be rewritten as

$$\begin{aligned}\mathcal{L}_{\mathrm{NJL}} &= \bar{q}\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m\right)q + 2G\langle\bar{q}q\rangle\bar{q}q - \frac{(M-m)^2}{4G}\\ &= \bar{q}\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} - M\right)q - \frac{(M-m)^2}{4G},\end{aligned} \tag{2}$$

where $M = m - 2G\langle\bar{q}q\rangle$ is the constituent quark mass. In the Hartree approximation, the self-energy of a quark is generated by the local four-fermion interaction. After some calculation, we can write

$$\begin{aligned}M &= m + 2\mathrm{i}G\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{Tr}\,S(p)\\ &= m + 8N_{\mathrm{f}}N_{\mathrm{c}}\mathrm{i}G\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{M}{p^2-M^2}.\end{aligned} \tag{3}$$

Using the standard technique of thermal field theory, we could directly calculate the contribution of the quark loops and polarization diagrams at finite temperature T and finite chemical potential μ. Thus, we could get the gap equation with the variables of T and μ as follows:

$$\begin{aligned}&\frac{M-m}{G} - 4N_{\mathrm{c}}N_{\mathrm{f}}\\ &\quad\times\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{M}{E_p}\left(1 - n_p(T,\mu) - \bar{n}_p(T,\mu)\right) = 0,\end{aligned} \tag{4}$$

where $E_p = \sqrt{p^2+M^2}$, and $n_p(T,\mu)$ and $\bar{n}_p(T,\mu)$ are the Fermi occupation numbers of quarks and antiquarks, respectively, with

$$\begin{aligned}n_p(T,\mu) &= \frac{1}{\mathrm{e}^{(E_p-\mu)/T}+1},\\ \bar{n}_p(T,\mu) &= \frac{1}{\mathrm{e}^{(E_p+\mu)/T}+1}.\end{aligned} \tag{5}$$

If we set $T = 0$, $\mu = 0$, (4) will share the same form as (3) after integration over p_0.

Solving the gap equation leads to the possible physical states of a system. To obtain the stable state which holds the lowest energy, one should conventionally compare the thermodynamical potentials corresponding to the solutions. In the NJL model, it is not difficult to obtain the thermodynamical potential Ω from the Lagrangian by applying the standard technique of thermal field theory.

For a system with volume V, temperature T and chemical potential μ, we could define the thermodynamical potential by

$$\Omega(T,\mu) = -\frac{T}{V}\ln\mathrm{Tr}\left[\exp\left(-\frac{1}{T}\int \mathrm{d}^3x\left(\mathcal{H}-\mu q^\dagger q\right)\right)\right], \quad (6)$$

where $\mathcal{H}$ is the Hamiltonian corresponding to the Lagrangian $\mathcal{L}$.

In the conventional way, as we take only the scalar condensate into account in the NJL model, we have

$$\mathcal{L} + \mu q^\dagger q = \bar{q}\left(\mathrm{i}\gamma^\mu\partial_\mu - M\right)q + \mu q^\dagger q - G\phi^2,$$

where $\phi = \langle\bar{q}q\rangle$ and $M = m - 2G\phi$. The thermodynamical potential could then be written

$$\Omega(T,\mu;M) = \Omega_M(T,\mu) + \frac{(M-m)^2}{4G}, \quad (7)$$

in which

$$\begin{aligned}\Omega_M(T,\mu) = -2N_\mathrm{c}N_\mathrm{f}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\Bigg\{&E_p \\ &+ T\ln\left[1+\exp\left(-\frac{1}{T}(E_p-\mu)\right)\right] \\ &+ T\ln\left[1+\exp\left(-\frac{1}{T}(E_p+\mu)\right)\right]\Bigg\}. \quad (8)\end{aligned}$$

It is well known that the stable physical state is the one corresponds to the global minimum of the thermodynamical potential, i.e., the one built upon the smallest Ω determined by the conditions $\frac{\partial\Omega}{\partial M}=0$ and $\frac{\partial^2\Omega}{\partial M^2}>0$. It is easy to see that the gap equations, (4) and (5), are the result of $\frac{\partial\Omega}{\partial M}=0$ for the corresponding Ω.

3 Characteristics of the chiral susceptibility in the NJL model

It has been shown that (7) and (8) represent the explicit form of the thermodynamical potential in the NJL model. From these equations we get the gap equation by evaluating the first order derivative of the Ω over the M and setting it zero, so that M is the solution of the gap equation. It explicitly reads

$$\begin{aligned}\frac{\partial\Omega}{\partial M} = \frac{M-m}{G} &- 4N_\mathrm{c}N_\mathrm{f}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{M}{E_p}\left[1-n_p(T,\mu)-\bar{n}_p(T,\mu)\right] \\ &\equiv 0. \quad (9)\end{aligned}$$

Furthermore, we get the explicit form of the second order derivative $\frac{\partial^2\Omega}{\partial M^2}$ as follows:

$$\begin{aligned}\frac{\partial^2\Omega}{\partial M^2}\bigg|_{\frac{\partial\Omega}{\partial M}=0} = \frac{1}{G} - 4N_\mathrm{c}N_\mathrm{f}\frac{\partial}{\partial M}\Bigg\{\int\frac{\mathrm{d}^3p}{(2\pi)^3} \\ \times\frac{M}{E_p}\left[1-n_p(T,\mu)-\bar{n}_p(T,\mu)\right]\Bigg\}. \quad (10)\end{aligned}$$

Taking the first order derivative of the gap equation over m, we obtain the expression of the chiral susceptibility χ:

$$\begin{aligned}\chi &= \frac{\partial M}{\partial m} \\ &= \frac{1}{1-4GN_\mathrm{c}N_\mathrm{f}\frac{\partial}{\partial M}\{\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{M}{E_p}[1-n_p(T,\mu)-\bar{n}_p(T,\mu)]\}}. \quad (11)\end{aligned}$$

From (10) and (11), we can easily find that the chiral susceptibility has a simple relation with $\frac{\partial^2\Omega}{\partial M^2}|_{\frac{\partial\Omega}{\partial M}=0}$; it reads

$$\chi = \frac{1}{G\frac{\partial^2\Omega}{\partial M^2}|_{\frac{\partial\Omega}{\partial M}=0}}. \quad (12)$$

Based on mathematical principles, we know that the second order derivative of a function determines the concavo–convexity of the function. For thermodynamical potential, the concavo–convexity shows the stability of the solutions of the gap equation. From (3) in the chiral limit (with $m\equiv 0$), one may notice apparently that there definitely exists a solution $M\equiv 0$, which corresponds to the phase with chiral symmetry and is usually referred to as the Wigner solution. There may also exist non-zero solution(s), corresponding to the chiral symmetry broken phase, being known as Nambu solution(s). Then, if $\frac{\partial^2\Omega}{\partial M^2}>0$ ($\frac{\partial\Omega}{\partial M}=0$), the function of the thermodynamical potential is concave, the state is stable, or at least, metastable. If $\frac{\partial^2\Omega}{\partial M^2}<0$ ($\frac{\partial\Omega}{\partial M}=0$), the function $\Omega(M)$ is convex, and the state associated with it is unstable. The point with $\frac{\partial^2\Omega}{\partial M^2}=0$ is the inflection of the function $\Omega(M)$. With the relation (12), one recognizes that the singularity of the chiral susceptibility (with $\frac{\partial^2\Omega}{\partial M^2}=0$) corresponds to the inflection point of the thermodynamical potential. Concerning the chiral susceptibility (χ_W) of the Wigner state, one recognizes that the singularity and the violation of the positivity of the chiral susceptibility indicates the possibility of a chiral phase transition, since χ_W being positive means that the Wigner state is stable and the system is in the chiral symmetry phase, and the negative χ_W

makes it manifest that the Wigner state is not stable; the system should then be in the dynamical chiral symmetry broken phase. Therefore, the chiral susceptibility could be a significant quantity as we identify the dynamical chiral phase transition, especially in the situation that people have difficulty in getting the explicit expression of the thermodynamical potential.

In the above discussion, we have not made any approximation, such as the chiral limit or the zero temperature limit. The obtained relation between the chiral susceptibility and the thermodynamical potential is general in the NJL model. In some other framework of non-perturbative approaches of QCD, for example, the Dyson–Schwinger equation approach, where one has the difficulty of getting the explicit expression of the thermodynamical potential, the method of analyzing the thermodynamical potential is thus not practical. However, calculating and analyzing the chiral susceptibilities *are* still available (for instance, Ref. [23] has done so successfully in the framework of the DSE of massless QCD). It makes manifest that analyzing the chiral susceptibilities may be a useful method to study the stability of states and the possibility of a chiral phase transition.

Besides the above mentioned relation between the chiral susceptibility and the thermodynamical potential, with which one can analyze the stability of the state, one may also predict the generation of new state(s) with the chiral susceptibility, since its divergence point just relates to the furcation point of the solutions of the gap equation in the NJL model. The above discussion has shown that the chiral susceptibility of the Wigner solution in the NJL model can be written as

$$\chi_W = \frac{1}{1-4GN_\mathrm{c}N_\mathrm{f}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p}[1-n-\bar{n}]}, \tag{13}$$

where $n = n_p\ (M = 0)$ and $\bar{n} = \bar{n}_p\ (M = 0)$. It shows apparently that the chiral susceptibility is positive–negative divergent at a particular chemical potential. To discuss the relation between the chiral susceptibility and the generation of new state(s), we take the functional derivative of the gap equation evaluated at $M = 0$. It gives

$$\delta M\left\{1-4GN_\mathrm{c}N_\mathrm{f}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p}[1-n-\bar{n}]\right\} = 0. \tag{14}$$

Taking advantage of the standard bifurcation theory (see for example Refs. [105–107]), one can obtain

$$1-4GN_\mathrm{c}N_\mathrm{f}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p}[1-n-\bar{n}] = 0. \tag{15}$$

It is evident that such a condition is just that for the chiral susceptibility of the Wigner solution to be divergent. Therefore the critical chemical potential for the chiral susceptibility of the Wigner solution to be divergent is also the one for the furcation of Wigner solutions to appear (or for more solutions to emerge). It indicates that analyzing the chiral susceptibilities can help us to study the structure of solutions of the gap equation, especially when we have difficulty in getting information of how many solutions exist for the gap equation.

4 Numerical verification in NJL model

To illustrate the validity and the consistency of the roles of the chiral susceptibility discussed analytically in the previous section, we firstly study the solutions of the gap equation and their chiral susceptibilities in the NJL model.

We constrain ourselves at present to the case of zero temperature and finite chemical potential. After some calculation from (4) and (5), the gap equations could be written explicitly as

$$\frac{M-m}{G} = 8N_\mathrm{c}\int_0^\Lambda\frac{p^2\,\mathrm{d}p}{2\pi^2}\frac{M}{E_p}, \quad \text{for } \mu < M, \tag{16}$$

$$\frac{M-m}{G} = 8N_\mathrm{c}\int_{k_\mathrm{f}}^\Lambda\frac{p^2\,\mathrm{d}p}{2\pi^2}\frac{M}{E_p}, \quad \text{for } \mu > M, \tag{17}$$

where m is the current quark mass, $\mu = \sqrt{M^2+k_\mathrm{f}^2}$ with k_f being the Fermi momentum, and Λ is the cut-off of the three-momentum.

Thus the chiral susceptibility for $\mu > M$ can be written as

$$\chi = \frac{\partial M}{\partial m} = \frac{1}{1-8N_\mathrm{c}G\frac{\partial^2}{\partial M^2}\int_{k_\mathrm{f}}^\Lambda\frac{p^2\,\mathrm{d}p}{2\pi^2}E_p - 8N_\mathrm{c}\mu G\frac{\partial^2}{\partial M^2}\int_{k_\mathrm{f}}^\Lambda\frac{p^2\,\mathrm{d}p}{2\pi^2}}, \tag{18}$$

and χ for $\mu < M$ is a constant.

4.1 In the chiral limit

We concentrate ourselves at present to the chiral limit, i.e., setting $m = 0$ in (16) and (17). To solve these gap equations, we take for the parameters $\Lambda = 587.90$ MeV and $G\Lambda^2 = 2.44$, with which the experimental data of the mass and the decay constant of the pion are well reproduced [33]. The solutions obtained for the gap equations are illustrated in Fig. 1.

The figure shows evidently that, at zero chemical potential, there exist three solutions to the gap equation, the Wigner solution $M = 0$ MeV and the Nambu solutions $M = \pm 387.92$ MeV. With increasing chemical potential, the Nambu solutions and Wigner solution would take on

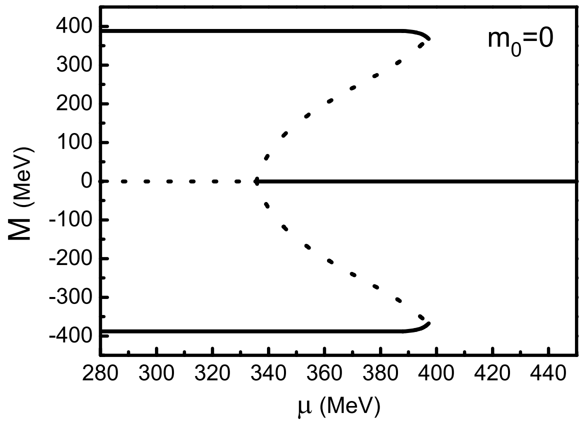

Fig. 1 Numerical solutions of the gap equation in terms of the chemical potential at the chiral limit and zero temperature in the NJL model (with parameters set $\Lambda = 587.9$ MeV and $G\Lambda^2 = 2.44$)

different behaviors. In detail: for the chemical potential $\mu \in (0, 335.59)$ MeV, there exists only one Wigner solution with $M = 0$ MeV. At the point $(335.59, 0)$, there emerges a trifurcation, one of which remains zero, the other two becoming positive and negative, respectively. As the chemical potential continuously increases, the zero solution maintains zero and the non-zero solutions change with the increasing absolute value. However, nothing special happens until $\mu = 387.92$ MeV. We know that, from this point, the solutions of (17) break the constraint $\mu < |M|$; thus, there are no longer constant solutions beyond this chemical potential. Moreover, just from such a μ, two more solutions to (17) appear. As the chemical potential takes a value in the region $\mu \in (387.92, 397.23)$ MeV, there are totally five solutions for (17); two of them are positive, one of them is zero and the other two are negative. As $\mu = 397.23$ MeV, the two positive solutions coalesce. It makes manifest that a bifurcation exists around $\mu \le 397.23$ MeV. The negative solutions have the same behavior as the positive ones. At a chemical potential larger than 397.234 MeV, there exists only one solution, $M \equiv 0$ (Wigner solution).

With the same parameters as taken to get Fig. 1, we calculate the chemical potential dependence of the chiral susceptibility of the states. The result obtained is illustrated in Fig. 2. The figure shows evidently that every appearance of the furcation in the solutions corresponds to the emergence of a singularity for the chiral susceptibility. Taking into account the result of the theory of functional analysis shown in (14) and (15), one can recognize that such a characteristic is quite natural. Thus, the variation behavior of the chiral susceptibility may make manifest the appearance of the furcation in the solutions, i.e., the generation and annihilation of the solution(s).

With (7) and (8), we have also evaluated the dependence of the thermodynamical potential on the chemical potential and the constituent (or dynamical) quark mass. The obtained result is illustrated in Fig. 3. From Fig. 3 one may notice that, as the chemical potential takes values through $\mu = 335.59$ MeV, the shape of the thermodynamical potential at $M = 0$ changes from a local maximum to a local minimum and the other minima correspond to $M \neq 0$ maintaining the global minima. As the chemical potential takes values through $\mu = 397.23$ MeV, the local minimum and local maximum of the thermodynamical potential at non-zero constituent mass coalesce and it holds a unique minimum at $M = 0$. This indicates that only the chiral symmetric phase exists. Comparing Figs. 3 and 2 we learn that the relation between the chiral susceptibility and the thermodynamical potential (see 12) is satisfied exactly.

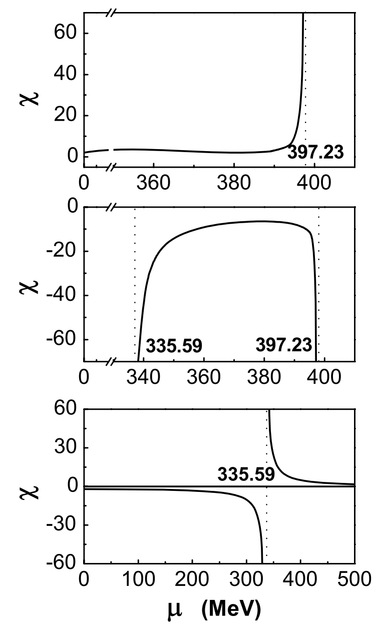

Fig. 2 Calculated chemical potential dependence of the chiral susceptibility (with the parameters taken the same as for Fig. 1); the *upper panel* is for the Nambu solutions, the *middle panel* for the non-zero Wigner solutions and the *lower panel* for the zero Wigner solution

4.2 Beyond the chiral limit

In the previous subsection, we discuss the solutions of the gap equation of the quark and the chiral susceptibility in the chiral limit in the NJL model. Now we go beyond the chiral limit. With the interaction parameters taken as those in the chiral limit and the current quark mass being set $m = 5.6$ MeV, which is consistent with the conventional choice, we solve (16) and (17).

At zero chemical potential, there exist three different solutions in such a model; this has carefully been discussed in Ref. [108] and implemented to study the relation of the explicit chiral symmetry breaking and the DCSB. In this paper, we extend this case to a finite chemical potential

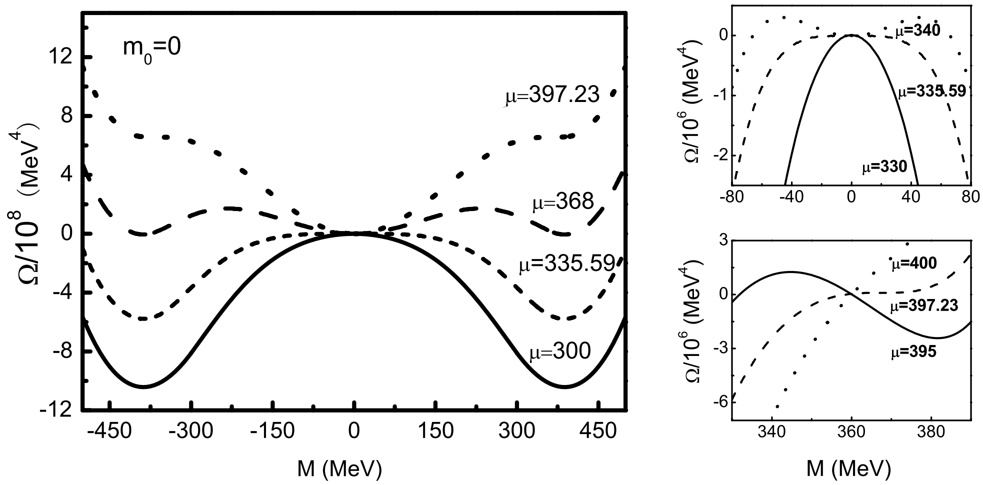

Fig. 3 Calculated constituent quark mass dependence of the thermodynamical potential in the chiral limit at several chemical potentials (*left panel*: general feature; *right panel*: detailed behavior of the evolution of the thermodynamical potential in two significant regions and around two special values of the chemical potential (with the thermodynamical potential at which $\frac{\partial^2 \Omega}{\partial M^2} = 0$ being normalized to zero, separately). The results are with the same interaction parameters as in Fig. 1)

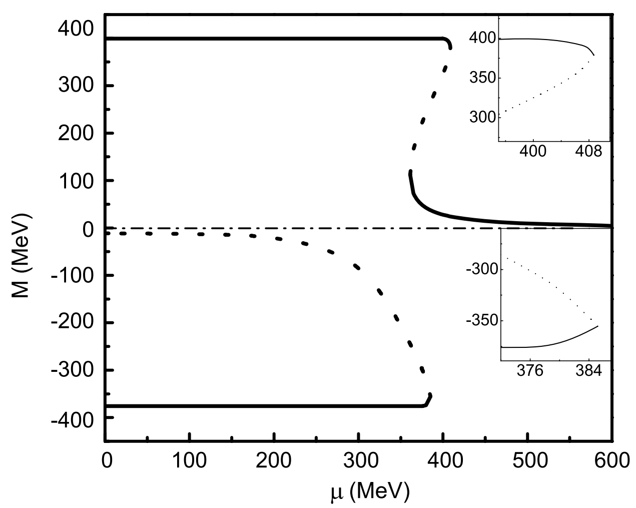

Fig. 4 Solutions of the gap equation beyond the chiral limit and at zero temperature as a function of the chemical potential (with the same parameters as those for Fig. 1 and a current quark mass $m = 5.6$ MeV)

and study the chemical potential dependence of these solutions. The obtained numerical result of the solutions against the chemical potential μ is illustrated in Fig. 4. The figure shows apparently that, at $\mu = 0$ MeV, (16) has the three solutions $M = 399.44$ MeV, -11.64 MeV and -375.89 MeV. At the point $(\mu, M) = (11.64, -11.64)$ MeV, the solution $M = -11.64$ MeV for (16) disappears and a solution for (17) starts exactly from this point. In the region $\mu \in (11.64, 361.08)$ MeV, there exist one solution for (17), whose absolute value increases with increasing μ, and two constant solutions for (16). As μ takes the value 361.08 MeV, another solution, with $M = 112.39$ MeV, emerges. Furthermore, from such a point, a bifurcation of the solutions appears, in which one increases with increasing μ and the other decreases. When μ reaches 375.89 MeV, the solution $M = -375.89$ MeV for (16) ends and a solution with the same value appears for (17). With increasing chemical potential, the value of such a negative solution of (17) increases. As the chemical potential reaches $\mu = 385.20$ MeV, the two negative solutions coalesce at $(\mu, M) = (385.20, -355.37)$ MeV. Then the negative solutions disappear, and there remains a bifurcation below that point. For the positive solution, the point $(\mu, M) = (399.44, 399.44)$ MeV is the end of the constant solution of (16), as well as the starting point of another solution for (17). With further increase of the chemical potential, this solution joins the upper solution generated at the point $(\mu, M) = (361.08, 112.39)$ MeV at the chemical potential $\mu = 408.70$ MeV. From this point, there is only one solution for (16) and (17) (in fact, only for (17)), and it gradually approaches the current quark mass when the chemical potential increases to positive infinite.

As defined above, we could get the explicit form of the chiral susceptibility from the gap equation, and it takes the same form as (18). If we require χ to be divergent and combine such a requirement with (18), we could get three solutions for the chemical potential μ. The results obtained are exactly the points for the bifurcations to appear in the solutions. The detailed numerical results of the dependence of the chiral susceptibility on the chemical potential is displayed in Fig. 5. Once again, we show that the singularity of the chiral susceptibility could help us to find the furcation of the solutions.

In addition, it may be interesting to discuss how the solutions beyond the chiral limit evolve from those in the chiral limit. In Ref. [108], some of us and collaborators show that there exists a convergence radius for the current quark mass, within which the dynamical (constituent) mass function can be expanded as a series in terms of the current mass. Then, if the current quark mass is small, the $m\chi$ could be a quite good approximation for the contribution of the current mass effect. Since the chiral susceptibility χ is positive for the Nambu solutions and negative for the Wigner solutions if the chemical potential is less than the corresponding critical value, we can understand that the positive and negative Nambu solution beyond the chiral limit come from the positive and negative Nambu solution in the chiral limit with increasing values, respectively. The Wigner solutions beyond the chiral limit arise from the Wigner solutions in the chiral limit with decreasing value, and they separate into two distinct branches due to the divergence of the chiral susceptibility.

With (6) and (7) and the parameters used above, we have also evaluated the thermodynamical potential as a function of the constituent quark mass and the chemical potentials in the case of going beyond the chiral limit in the NJL model. The obtained results are displayed in Fig. 6. Figure 6 shows evidently that when the chemical potential is small, the thermodynamical potential holds three extremes. Two of them are minima and correspond to the positive and negative Nambu solution, respectively; the other one is a maximum and corresponds to the Wigner solution. It indicates apparently that the Nambu solutions are stable states and the Wigner solution is an unstable state, so that the system is in chiral symmetry dynamically broken phase. As μ takes the value 361.08 MeV, an inflection emerges at $M = 112.39$ MeV which separates the convex part of the thermodynamical potential into a concave part and a convex part, and a local minimum appears at $M \approx 5.6$ MeV. Recalling the solutions of the gap equation shown in Fig. 4, we know that such a point corresponds to the one for a bifurcation to appear. When the chemical potential reaches 385.20 MeV, the local minimum at $M \approx 5.6$ MeV becomes the global minimum, and the original global minimum at $M = 399.44$ MeV changes to a local minimum; also, the local minimum corresponding to negative M disappears. In turn, the dynamical chiral symmetry can be restored. As the chemical potential $\mu = 408.70$ MeV, the local minimum disappears. Then there exists only one minimum at $M \approx 5.6$ MeV as $\mu > 408.70$ MeV, and thus only the dynamical chiral symmetric phase exists.

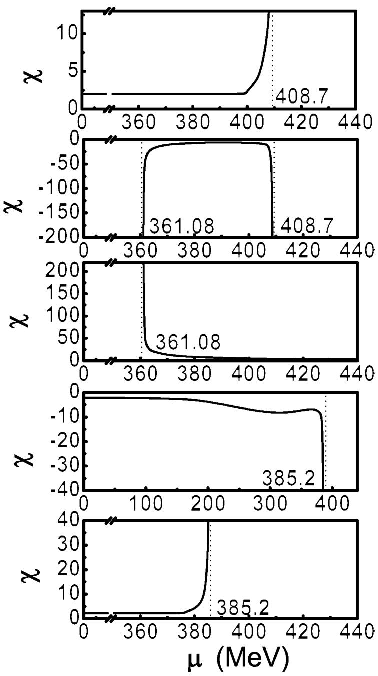

Fig. 5 Calculated dependence of the chiral susceptibility on the chemical potential beyond the chiral limit (with the parameters being taken the same as for Fig. 4). The *panels from top to bottom* correspond to the solution illustrated in Fig. 4 from the uppermost to the lowermost sequentially

4.3 The chiral phase transition

The above numerical results and discussions apparently make manifest that the characteristic of the chiral susceptibility represents the structure of the solutions of the gap equation and the behavior of the thermodynamical potential in the cases of not only working in the chiral limit but also beyond the chiral limit in the NJL model. We would now

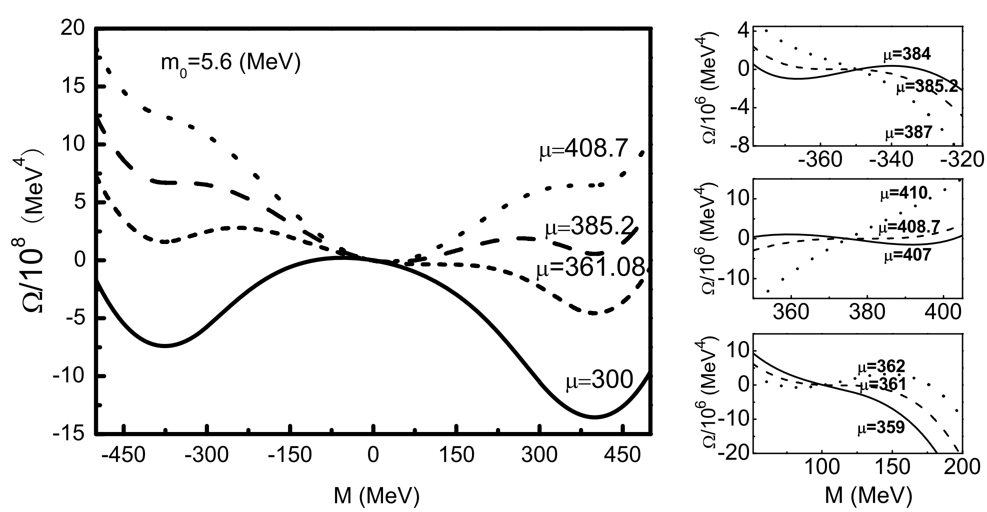

Fig. 6 Calculated constituent quark mass dependence of the thermodynamical potential beyond the chiral limit at several chemical potentials (*left panel*: general feature; *right panel*: detailed behavior of the evolution of the thermodynamical potential in three significant regions and around three particular values of the chemical potential (with the thermodynamical potential at which $\frac{\partial^2\Omega}{\partial M^2} = 0$ is normalized to zero, separately). The results are for the same interaction parameters as for Fig. 4)

like to discuss further the role of the chiral susceptibility in the chiral phase transition. With the general approach in analyzing the phase transition (comparing the difference of thermodynamical potentials between the DCSB phase and the chiral symmetric one in the present case), we learn that the value $\mu_c = 368$ MeV, 385.2 MeV can be regarded as the critical chemical potential for the chiral phase transition (from DCSB to the chiral symmetric one) to take place in the chiral limit and beyond the chiral limit, respectively. At first sight, the behavior of the chiral susceptibility could not determine the critical chemical potential, even though it gives the same result in the case of going beyond the chiral limit with a current quark mass $m_0 = 5.6$ MeV. However, we would see that the chiral susceptibility also plays a significant role in identifying a first order phase transition at finite chemical potential. Looking at the numerical results, we may notice that, in the case of the chiral limit, as the chemical potential is in the region (335.59, 397.23) MeV, the positivity of the chiral susceptibility in the Wigner phase and that in the Nambu phase indicate that both the Wigner phase and the Nambu phase are stable (or one is stable, and the other metastable) simultaneously. The divergence of the chiral susceptibility in the Wigner phase at $\mu_W^c = 335.59$ MeV and that in the Nambu phase at $\mu_N^c = 397.23$ MeV indicate that there exists another state which is unstable. In the case of going beyond the chiral limit, when the current quark mass takes the value $m = 5.6$ MeV, for example, the critical point $\mu_c = 385.2$ MeV is definitely in the region (361.08, 408.7) MeV wherein the Wigner phase and Nambu phase are both stable. The coexistence of two stable phases demonstrates the possibility of a first order phase transition, and the phase transition actually occurs in this region based on the competition of the two phases in the general point of view of phase transitions. Furthermore, since the region $\mu_N^c - \mu_W^c$, in which the phase transition takes place, is much smaller than the constituent quark mass scale (i.e., the scale of the dynamical chiral symmetry breaking), we can estimate the critical chemical potential by the interpolation $\mu_c' = \frac{\mu_W^c + \mu_N^c}{2}$, which gives $\mu_c' = 366.41$ MeV and 384.89 MeV for the case of the chiral limit, and going beyond the chiral limit, respectively. Comparing these values with those obtained by analyzing the thermodynamical potential, we find that the difference is less than 2 MeV.

The above analysis indicates that, even though the critical chemical potential cannot exactly be determined by studying the chiral susceptibility, the divergences of the chiral susceptibility of both the Wigner phase and the Nambu phase identify at least a region in which the stable (metastable) Wigner phase and metastable (stable) Nambu phase exist simultaneously. The phase transition takes place definitely in this region and the critical chemical potential can be estimated with $\mu_c' = \frac{\mu_W^c + \mu_N^c}{2}$ with a quite high precision. Comparing the result in the chiral limit with that beyond the chiral limit in the NJL model, we notice that the region in which the chiral phase transition occurs becomes narrower and the error of the fixed critical chemical potential (compared with that given by analyzing the thermodynamical potential) becomes smaller with the increase of the current quark mass. It can be inferred that the phase transition region also becomes narrower with the increase of temperature and goes continuously to zero at the end point of the first order phase transition. Related investigations in the μ–T plane are in progress.

5 Summary and remarks

In summary: we give in this paper a general relation between the chiral susceptibility and the thermodynamical potential and an explicit relation between the chiral susceptibility and the condition for furcation to emerge in the Wigner solution(s) in NJL model. Both analytical and numerical results make manifest that the singularity of the chiral susceptibility corresponds to the point for the furcation to appear in the solution(s) as well as for the concavo–convexity of the thermodynamical potential to change. It indicates that the chiral susceptibility χ is a quantity able to characterize the stability of the possible states and to identify the critical chemical potential; this is taken as the average of the ones for the chiral susceptibilities requiring both the Wigner solutions and Nambu solutions to be positively divergent to a quite high precision. Analyzing the chiral susceptibilities of both the Nambu solutions and the Wigner solutions of the gap equation simultaneously is thus an efficient method in studying the chiral phase transition in the NJL model.

It is well known that QCD phases and the phase transition depend not only on the medium effects, such as temperature, density and finite size of the matter, but also on the intrinsic characteristics of the quarks, such as the isospin, the color–flavor structure, the running coupling strength, the current mass and so on. Due to the complicated non-perturbative property of QCD in the low energy region, one does not yet have quantitative knowledge of the complete QCD phase structure and its transition. In the NJL model, one can relatively easily get all the solutions of the gap equation and obtain the thermodynamical potential, and then study the chiral phase transition. As mentioned above, in other approaches, such as the DSE of QCD, one always has difficulty in knowing how many solutions exist for the gap equations and in finding the solutions completely. The question of how to obtain the exact thermodynamical potential is still under investigation. Studying the chiral phase transition with the thermodynamical potential is thus not practical. However, the chiral susceptibility is a well-defined quantity in the DSE. Pushing the above discussion forward, we may infer that, if the relation $\chi \propto \frac{1}{\frac{\partial^2 \Omega}{\partial M^2}}$ does work well in the DSE of QCD, it will be helpful in studying the QCD phase transition. For example, in the chiral limit, we know of the existence of a Wigner solution. If we calculate the chiral susceptibility of the Wigner solution and find the divergency of χ, we could know that other solution(s) would be generated at the divergence point (such an analysis has been performed in Ref. [23]). Then by varying the relevant argument(s) of the points around it, one could easily find the track of other solutions. By evaluating the chiral susceptibility of the new solution, we could find where this solution ends and other solutions start, which is known to be the Nambu solution. A similar analysis could be taken beyond the chiral limit. The successful search for the Wigner solution and its conjunction and disappearance together with the negative Nambu solution of the DSE of QCD beyond the chiral limit [108] is just simulated by such an analysis (one usually believes that a Wigner solution does not exist in the case of going beyond the chiral limit, since the chiral symmetry has been broken explicitly. In fact, there still exists dynamical chiral symmetry and its spontaneous breaking if the current quark mass is not large enough [108]. Besides, Pennington and collaborators have not only confirmed the existence of the Wigner solution we gave, but also found another solution [109]). We thus propose that starting from analyzing the variation behavior of the chiral susceptibility of the solution(s), which could be obtained easily, with respect to each of the ingredients influencing the phases can help us to study the QCD phase transitions, especially in the case that we could not easily know how many solutions exist for the gap equations and have difficulty in finding all the solutions and in determining the thermodynamical potential. Related investigations in the formalism of the Dyson–Schwinger equation are in progress.

Acknowledgements This work was supported by the National Natural Science Foundation of China under the Grant Nos. 10425521, 10675007 and 10705002, the National Fund for Fostering Talents of Basic Science (NFFTBS) with contract No. J0630311, the Major State Basic Research Development Program under Contract No. G2007CB815000, the Key Grant Project of Chinese Ministry of Education under contract No. 305001. Helpful discussions with Professor D.P. Li are acknowledged with thanks.

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